

## Wave-number dependence of the transitions between traveling and standing vortex waves and their mixed states in the Taylor-Couette system

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Previous numerical investigations of the stability and bifurcation properties of different nonlinear combination structures of spiral vortices in a counter-rotating Taylor-Couette system that were done for fixed axial wavelengths are supplemented by exploring the dependence of the vortex phenomena waves on their wavelength. This yields information about the experimental and numerical accessibility of the various bifurcation scenarios. Also backward bifurcating standing waves with oscillating amplitudes of the constituent traveling waves are found.

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Recently the stability exchange between traveling waves (TWs) [1] and standing waves (SWs) of spiral vortices in the Taylor-Couette system has been investigated by full numerical simulations and a coupled amplitude equation approximation [2]. TWs and SWs have a common onset as a result of a primary, symmetry degenerate oscillatory bifurcation. The SW solution is a nonlinear superposition of mirror symmetric, oppositely propagating TWs with equal amplitudes. At onset either the TW or the SW solution is stable [3,4]. Then, at larger driving there is a secondary bifurcation that leads to a stability exchange between the two solutions. This exchange is mediated by mixed patterns that establish in solution space a connection between a pure TW and a pure SW. The mixed structures consist of a superposition of oppositely propagating TWs with temporally constant, nonequal amplitudes. The TWs investigated in Ref. [2] are initially stable while the SWs gain stability later on.

There is a second variety of mixed states in which the TW amplitudes oscillate in time in counterphase. This stable solution bifurcates out of the SW at even higher driving rates via a Hopf bifurcation [5] in which the aforementioned SWs lose their stability. These results have been found by full numerical simulations of the vortex flow in a Taylor-Couette system [6,7] with counter-rotating cylinders of radius ratio  $\eta=0.5$  with methods described in Ref. [8]. The calculations were done for a fixed axial wavelength  $\lambda$  by imposing axially periodic boundary conditions [9].

Here we investigate and show how stability, bifurcation properties, and the spatiotemporal behavior of the aforementioned structures change with  $\lambda$ . Thus, this report provides information for future simulations and experiments with finite length setups and, say, nonrotating lids that close the annular gap between the cylinders at the ends: Since the height of the system influences the wavelength of the vortex structures and with it their properties the prior knowledge of their  $\lambda$  dependence is of significant interest.

*Structures.* The following structures have been investigated. (i) Forward bifurcating TWs consisting of left-handed spiral vortices (L-SPI) or of right-handed spiral vortices (R-SPI) that are mirror images of each other. L-SPI (R-SPI)

travel in the annulus between the two cylinders axially into (opposite to) the direction of the rotation frequency vector of the inner one, i.e., in our notation upwards (downwards) [8]. (ii) Forward bifurcating SWs that consist of an equal-amplitude nonlinear combination of L-SPI and R-SPI. These SWs are called ribbons (RIBs) in the Taylor-Couette literature [10,11]. (iii) So-called cross-spirals (CR-SPI), i.e., combinations of L-SPI and R-SPI with different stationary amplitudes. They provide a stability transferring connection between TW and SW solution branches [2,6]. And, finally, (iv) oscillating cross-spirals (O-CR-SPI). Therein, the amplitudes of the TW constituents of the SW, i.e., the amplitudes of L-SPI and the R-SPI oscillate in counterphase around a common mean [5]. The vortex structures (i)–(iv) are axially and azimuthally periodic with axial wave number  $k=2\pi/\lambda$  and azimuthal wave number  $M=2$  in our case.

*Control and order parameters.* The control parameters are the Reynolds numbers  $R_1 > 0$  and  $R_2 < 0$  defined by the rotational velocities of the inner and outer cylinder, respectively. As order parameters we use the amplitudes

$$A(t) = u_{2,1}(t), \quad B(t) = u_{2,-1}(t) \quad (1)$$

of the dominant critical modes of the radial velocity  $u$  at midgap in the double Fourier decomposition in azimuthal and axial direction. In Eq. (1) the indices  $m=2$  and  $n=\pm 1$  identify azimuthal and axial modes, respectively. Note that for SPI, CR-SPI, and RIB structures investigated here the moduli in Eq. (1) are constant. On the other hand, in O-CR-SPI the moduli  $|A(t)|$  and  $|B(t)|$  oscillate in counterphase around a common mean. Therefore we use the difference of the squared moduli  $D(t) = [|A(t)|^2 - |B(t)|^2]/2$  and its oscillation amplitude  $\tilde{D}$  to describe the bifurcation from the RIB solution ( $\tilde{D}=0$ ) to O-CR-SPI ( $\tilde{D} \neq 0$ ).

*$\lambda$  dependence of the bifurcation scenario.* Figure 1 shows the  $\lambda$  dependence of the bifurcation thresholds  $R_1^0$  for  $M=1$  and  $M=2$  SPI and RIB and  $M=0$  Taylor vortex flow. These results were obtained from a linear stability analysis [12] of the basic circular Couette flow. For the two characteristic values  $R_2 = -540$  and  $-605$  shown there the  $M=2$  SPI and RIB have the lowest threshold for a wide range of  $0.8 \leq \lambda \leq 2.1$ .

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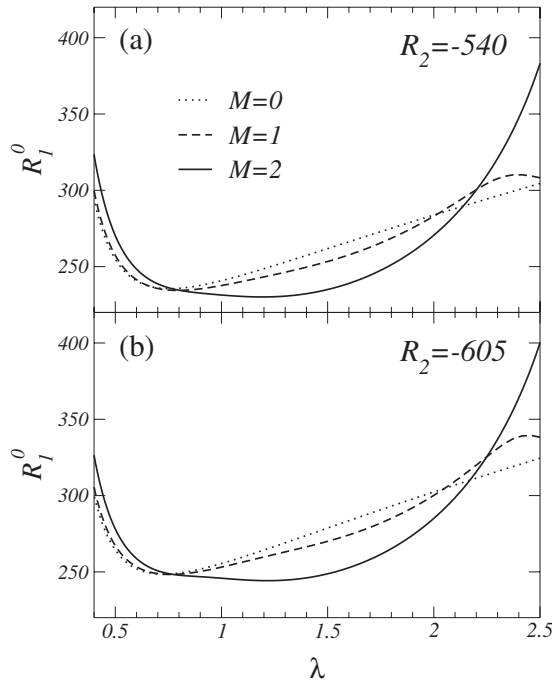


FIG. 1. Bifurcation thresholds  $R_1^0$  of  $M=0$  Taylor vortex flow and of SPI and RIB with azimuthal wave numbers  $M=1$  and  $M=2$  versus axial wavelength  $\lambda$  for different  $R_2$  as indicated.

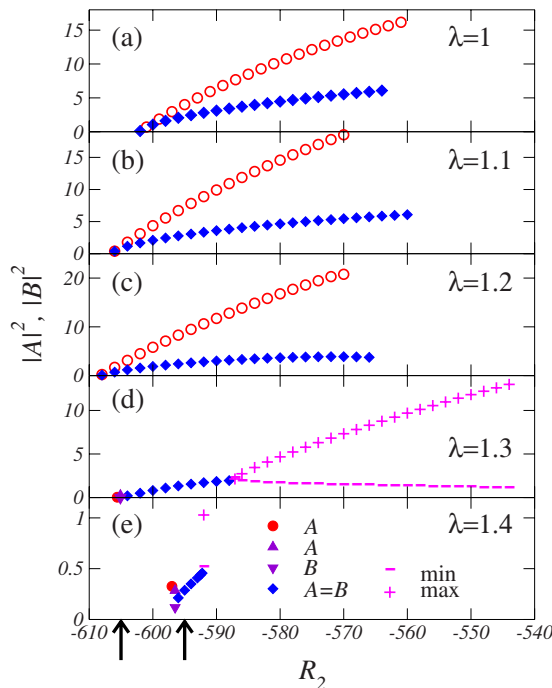


FIG. 2. (Color online) Influence of the axial wavelength  $\lambda$  on the bifurcation behavior for fixed  $R_1=240$ . Solid (open) symbols denote stable (unstable) structures. Symbols show L-SPI (red circles,  $A \neq 0, B=0$ ), L-CR-SPI (violet triangles,  $|A| > |B|$ ), RIB (blue diamonds,  $A=B$ ), and stable O-CR-SPI (magenta). In the latter  $|A(t)|$  and  $|B(t)|$  oscillate between the + and - symbols. In (d) and (e) only stable structures are shown. The arrows refer to the  $R_2$  values of Fig. 3.

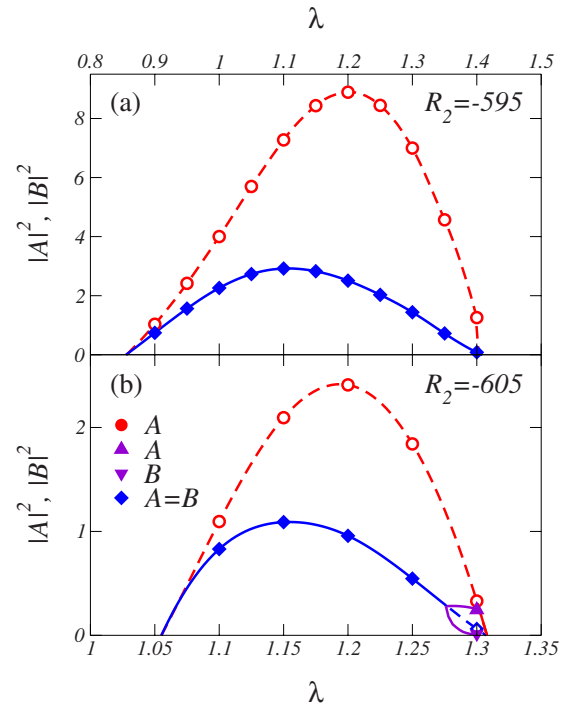


FIG. 3. (Color online) Bifurcation diagrams of L-SPI (red circles,  $A \neq 0, B=0$ ), RIB (blue diamonds,  $A=B$ ), and L-CR-SPI (violet triangles,  $|A| > |B|$ ) as a function of the axial wavelength  $\lambda$  for fixed  $R_1=240$  and the two values of  $R_2$  that are indicated by arrows in Fig. 2. Solid (open) symbols denote calculated stable (unstable) structures. Full (dashed) lines for stable (unstable) solutions branches result from a spline interpolation. However, the violet L-CR-SPI branches are schematic.

In Fig. 2 we show for different  $\lambda$  bifurcation diagrams as functions of  $R_2$  for a fixed  $R_1=240$ . For  $1 \leq \lambda \leq 1.2$ , figure parts (a)–(c), the bifurcation properties are quite similar: SPI and RIB bifurcate supercritically out of the circular Couette flow, SPI (RIB) are unstable (stable) at onset, and there is no stability exchange in the range of  $R_2$  of Fig. 2. By contrast, for  $\lambda=1.3$  and  $1.4$ , in (d) and (e), respectively, there are different interesting stability exchanges. Here, L-SPI ( $A \neq 0, B=0$ ) and R-SPI ( $A=0, B \neq 0$ ) are initially stable while the RIB state ( $A=B$ ) is initially unstable. But RIB gain stability almost immediately thereafter: the stability transfer from L-SPI or R-SPI to RIB is mediated within a very small interval by the L-CR-SPI ( $|A| > |B|$ ) or the R-CR-SPI ( $|B| > |A|$ ) solution, respectively [2]. For larger driving, the RIB lose stability again, when stable oscillating structures, O-CR-SPI, appear via a Hopf bifurcation [5]. Note, however, that O-CR-SPI bifurcate forward for  $\lambda=1.3$  but backward for  $\lambda=1.4$ , see details further below.

In Fig. 3 we show bifurcation diagrams as a function of  $\lambda$  for  $R_1=240$  and two different  $R_2$  indicated by arrows in Fig. 2. In the case of  $R_2=-595$  (right arrow in Fig. 2) SPI are unstable and RIB are stable for all  $\lambda$ . For  $R_2=-605$  (left arrow in Fig. 2), on the other hand, this stability situation—RIB are stable and SPI are unstable—applies only as long as  $\lambda \leq 1.25$ : Then, with increasing  $\lambda$ , a stability exchange between RIB and SPI via CR-SPI occurs that is reflected also at the very left end of the bifurcation diagram in Fig. 2(d). So,

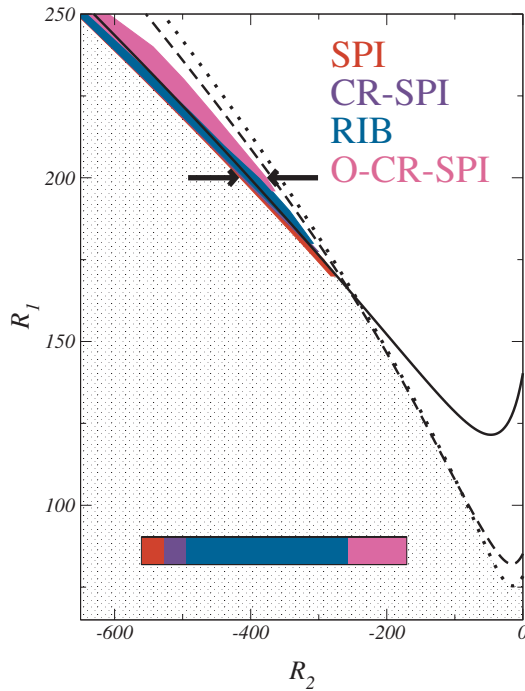


FIG. 4. (Color). Phasediagram of stable SPI, CR-SPI, RIB, and O-CR-SPI. In the colored areas these vortex structures (as indicated by the color code) with azimuthal wave number  $M=2$  and axial wavelength  $\lambda=1.3$  are stable. The white area has not been investigated in this work. The basic state of circular Couette flow is stable in the dotted region: Lines are marginal stability boundaries of circular Couette flow against growth of vortex flow with  $\lambda=1.3$  and azimuthal wave numbers  $M=2$  (full),  $M=1$  (dashed), and  $M=0$  (dotted). The bar shows a blow-up of the region between the black arrows.

the interesting stability exchange between TWs and SWs occurs in a rather narrow wave number band around  $\lambda \approx 1.3$ .

In Figs. 2 and 3, we showed the case where stability is transferred from L-SPI to RIB. The symmetry degenerated situation where stability is transferred from R-SPI to RIB via R-CR-SPI is obtained by exchanging (the symbols for)  $A$  and  $B$  in these figures.

*Phase diagram for  $\lambda=1.3$ .* In view of the above discussed stability exchange process we take a more detailed look at the case  $\lambda=1.3$  for which previous calculations have been done only at the two Reynolds numbers  $R_1=200$  and  $240$  [2,5,13]. To that end we provide in Fig. 4 the phase diagram of the stable, aforementioned  $M=2$  vortex structures with fixed  $\lambda=1.3$  in the  $R_1$ - $R_2$ -parameter plane. Stable  $M=2$  SPI appear first via a primary forward bifurcation at the lower left border of the red stripe in Fig. 4 [14]. Then, for a fixed  $R_1 \geq 190$ , we have observed with increasing  $R_2$  always the same stability transfer sequence

$$\text{SPI} \rightarrow \text{CR-SPI} \rightarrow \text{RIB} \rightarrow \text{O-CR-SPI}.$$

For lower  $R_1$ , however, the existence range of stable  $M=2$  structures seems to be more and more confined from above by the appearance of  $M=1$  modes at the respective bifurcation threshold (dashed line): With decreasing  $R_1$  first the O-CR-SPI and then the RIB and CR-SPI areas are pinched

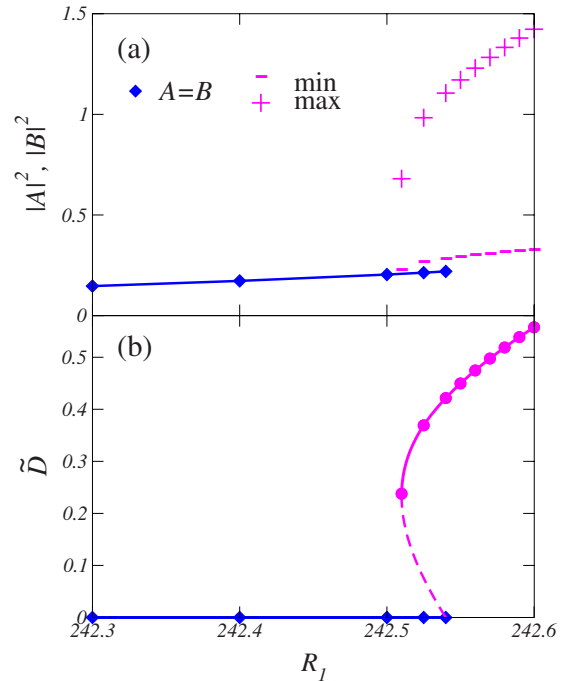


FIG. 5. (Color online). Backward bifurcation of O-CR-SPI (magenta circles and + and - signs) out of RIB (blue squares) for  $\lambda=1.4$  and  $R_2=-605$  as a function of  $R_1$ . In (a) squared amplitudes  $|A|^2, |B|^2$  are shown with symbols as explained in Fig. 2. The oscillation amplitude  $\tilde{D}$  of  $D=(|A|^2-|B|^2)/2$  is shown in (b). For RIB  $\tilde{D}=D=0$ . Full (dashed) lines denoting stable (unstable) solutions were obtained by spline interpolation.

off successively. Note that in all cases the CR-SPI stripe is extremely thin (cf. the blow-up bar in Fig. 4) whereas the O-CR-SPI area being quite large should facilitate a respective experimental observation.

*Backward bifurcating O-CR-SPI.* As noted already in the discussion related to Fig. 2(e), we have found for the first time backward bifurcating O-CR-SPI. In Fig. 5 we display for fixed  $\lambda=1.4$  and  $R_2=-605$  as a representative example the bifurcation properties of this new scenario. Figure 5(a) shows the squared moduli  $|A|^2$  and  $|B|^2$  as a function of  $R_1$ . Diamonds show stable RIB. They have obtained their stability from the SPI via a CR-SPI branch connection at smaller  $R_1$  outside the plot range of Fig. 5.

The + and - signs denote the maximal and minimal amplitudes, respectively, of the modes  $A$  and  $B$  that oscillate in counterphase in the O-CR-SPI. The hysteresis in the transition between stable RIB and stable O-CR-SPI is best visible with the order parameter  $\tilde{D}$  in Fig. 5(b).

*Conclusion.* Our results show that the mixed states of stationary CR-SPI and of O-CR-SPI should be observable in experimental setups or in finite lengths numerical simulations when the wavelength of these vortex structures lies in the interval of  $1.3 \leq \lambda \leq 1.4$ . Therein O-CR-SPI are stable in a wide range of control parameters. They bifurcate either forward or, as we have found here, backward out of the RIB state of standing waves. CR-SPI solutions, on the other hand, exist only in a rather small interval of control parameters.

- [1] Throughout this manuscript we use the following abbreviations. TW: traveling wave, SW: standing wave, SPI: spiral vortices, L-SPI: left-handed spiral vortices, R-SPI: right-handed spiral vortices, RIB: ribbon, CR-SPI: cross-spirals, O-CR-SPI: oscillating cross-spirals.
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- [14] The difference between the bifurcation threshold obtained from the finite-differences numerical simulation (left border of the red area in Fig. 4) and the full line obtained from a linear stability analysis of the circular Couette flow is due to the limited numerical resolution of the former. This difference increases with increasing  $R_1, |R_2|$ : For higher Reynolds numbers the vortex flow intensifies close to the inner cylinder. This uneven flow distribution is better captured by the very high spatial resolution used in the linear analysis than with the fixed homogeneous grids used in the nonlinear finite-differences calculations. In Fig. 4 the differences do not exceed 3–4 %.